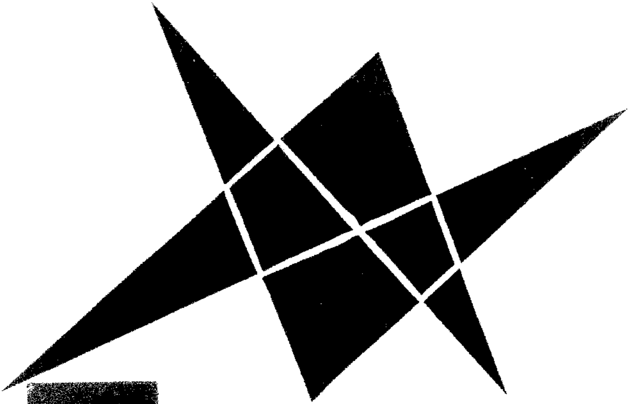
TOPICS IN MATHEMATICS
TRANSLATED FROM THE RUSSIAN

CONFIGURATION THEOREMS

B. I. ARGUNOV L. A. SKORNYAKOV

D. C. HEATH AND COMPANY



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TOPICS IN MATHEMATICS

Configuration Theorems

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Translated and adapted from the first Russian edition (1957) by

EDGAR E. ENOCHS *and* ROBERT B. BROWN

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PREFACE TO THE AMERICAN EDITION

THIS BOOKLET presents several important configuration theorems, along with their applications to the study of the properties of figures and to the solutions of several practical problems. In doing this, the authors introduce the reader to some fundamental concepts of projective geometry—central projection and ideal elements of space. Only the most elementary knowledge of plane and solid geometry is presupposed.

Chapters 2 and 3 are devoted to the two most important configuration theorems, the Pappus-Pascal theorem and that of Desargues. The chapters which follow present applications of these theorems. Chapter 6 touches upon the algebraic interpretation of configuration theorems and the general method of arriving at such theorems.

Students who wish to learn more about this subject should consult the bibliography given at the end of the booklet.

CONTENTS

Introduction

1. WHAT IS A CONFIGURATION THEOREM?

A *configuration*[1] *theorem* is a theorem that concerns a finite number of points and lines[2] and their incidence relationships. Usually a configuration theorem states that because several points all lie on a line or because several lines pass through one point, certain other points have to be situated on one line or some other lines pass through a common point.

The following are simple examples of configuration theorems:

1. *If A, B, C, and D are four different points such that A, B, and C are on one line and A, B, and D are on one line, then B, C, and D are on one line* (Fig. 1).

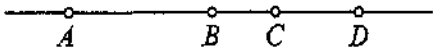

Fig. 1

2. *If a, b, c, and d are four different lines such that a, b, and c pass through a common point and a, b, and d pass through a common point, then b, c, and d also pass through a common point* (Fig. 2).

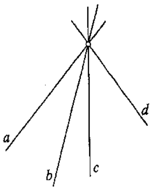

Fig. 2

Of the propositions studied in a school geometry course, those which deal with the notable points of a triangle[3] bear the closest resemblance to configuration theorems.

Some configuration theorems were known in antiquity. In more recent times they have formed the basis of one of the

[1] Configuration (Latin, *configuratio*), a figure made up of related elements.

[2] In this booklet a "line" is always a straight line infinitely prolonged in both directions.

[3] Some notable points of a triangle are the intersection point of the medians, the intersection point of the altitudes, and the centers of the circumscribed and inscribed circles.

most interesting branches of geometry—projective geometry. Projective geometry, in turn, forms the basis for descriptive geometry —the study of representing three-dimensional figures upon plane surfaces.

In the last decade algebraists have become interested in configuration theorems, as is partly explained in the last section of this booklet.

Configuration theorems can be applied successfully to the study of the properties of polygons and to the solution of problems concerning them. They are especially useful for solving construction problems with various restrictive conditions: for construction using only the straightedge, for construction in a bounded part of the plane, for constructions with inaccessible points, etc.

2. EXAMPLE OF A CONFIGURATION THEOREM

Draw a triangle ABC and choose a point O inside it. Extend the lines joining each of the vertices with O until they intersect the opposite sides of the triangle in points P, Q, and R (Fig. 3). Let U be the point of intersection of the lines AB and PQ, V the point of intersection of the lines AC and PR, and W the point of intersection of the lines BC and QR. If the points all fall within the confines of the drawing and the construction has been carried out accurately, U, V, and W will all appear to lie on one line. If another triangle and another point O are chosen, the result will be the same.

If it turns out that $BC \parallel QR$, then it will also happen that $UV \parallel QR$. If $BC \parallel QR$ and $AC \parallel PR$, then necessarily $AB \parallel PQ$, as will be shown later.

It would be hard to believe that these things happen by chance. Apparently there is conformity to some

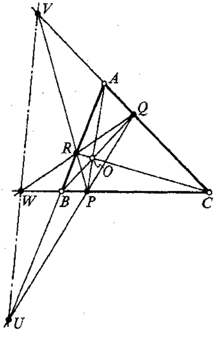

Fig. 3

law here; some theorem must be true. The conclusion of the theorem would be "the points *U, V,* and *W* all lie on one line." As mentioned above, this may be compared with the theorems about the notable points of a triangle encountered earlier in school. But in contrast to these theorems, there are no conditions on the size of the sides or angles in our theorem. In both the conditions and the conclusion, only the *relative position* of points and lines is mentioned. Such a theorem is called *configurational.*

The configuration theorem introduced above may be stated as follows:

If the vertices of the triangle PQR lie respectively on the sides of the triangle ABC with P, Q, and R on the sides opposite A, B, and C, respectively, and, furthermore, the three lines AP, BQ, and CR pass through a common point, and the lines AB and PQ, AC and PR, BC and QR intersect at the points U, V, and W, respectively, then the points U, V, and W all lie on one line.

In order to be able to prove configuration theorems, we need to acquaint ourselves with the operation of central projection and with the concept of "ideal" or "infinitely remote" points and lines.

1. Central Projection and Ideal Elements

3. CENTRAL PROJECTION IN THE PLANE; IDEAL ELEMENTS

Let there be given a line l and a point S not on the line (Fig. 4). If A is a point such that AS is not parallel to l, then the line SA intersects the line l in some point A'. We shall call this point the *projection* on l of the point A. Let m be the line passing through the point S parallel to l. Then it is clear that every point B in the plane not on the line m will have a projection B'. Thus, using the point S, we *map* the points of the plane onto the points of the line l. Such a mapping is called a *central projection*, and the point S, the *center of projection*.

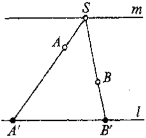

Fig. 4

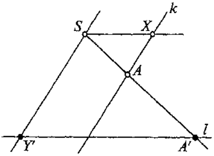

Fig. 5

If we choose any line k not passing through the point S, then the central projection maps the points of the line k onto the line l (Fig. 5). If $k \nparallel l$, then not all the points of the line k have projections. The point X will not have a projection on l if $SX \parallel l$. From Fig. 5, it is also easy to see that Y' on l is another exceptional point if $SY' \parallel k$; it is the only point of the line l which is not the projection of any point of the line k.

In order to deal with such exceptional points in the same way as the other ones, we shall agree that besides the ordinary points, each line will have one *ideal* point in addition to all of its other points. We shall also stipulate that parallel lines are to have a *common* ideal point.[1]

Now the point X in Fig. 5 will have a projection: its projection will be the ideal point of the line l. The point Y' will, in turn, be the

[1] In space, as well as in the plane, we agree that parallel lines have the same ideal point.

projection of the ideal point of the line k. Note that in case k and l are parallel, their common ideal point will be its own projection.

Now we notice the following: If the point A approaches the point X from the left (in Fig. 6 these positions of A are labeled A_1,

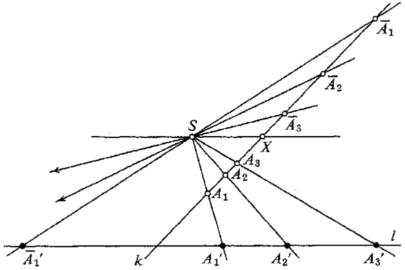

Fig. 6

A_2, A_3, etc.), then its projection moves farther and farther to the right along the line l. For this reason, an ideal point is also called "infinitely remote."

Furthermore, notice that as the point A approaches the point X from the right (in Fig. 6 these positions of A are labeled $\overline{A}_1, \overline{A}_2, \overline{A}_3$, etc.), its projection recedes to the left along the line l. Since we regard each line as having only one infinitely remote point, we may consider the line itself as "closed."

A line with an ideal point added is called a *projective line*. Let us agree to consider that the ideal points of all the lines in a plane lie on one *ideal* projective line, which we shall also call "infinitely remote." The plane with the addition of ideal points and the ideal line we shall call the *projective plane*.

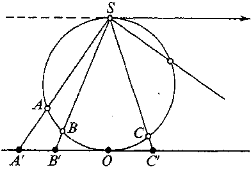

Fig. 7

Remark. A projective line is "similar" to a circle. In fact, a correspondence can be set up between points of the projective line and points of the circle

in such a manner that different points of the circle will correspond to different points of the projective line. The method of setting up this correspondence is clear from Fig. 7, if we add that the point O corresponds to itself and that S corresponds to the ideal point of the line.

4. ELIMINATION OF EXCEPTIONAL CASES

The introduction of ideal points enables us to include the exceptional cases of configuration theorems in a general formulation. In the case of the theorem considered in section 2, these exceptional cases arise if any of the lines defining the points U, V, and W are parallel. This means that the corresponding point would be ideal. In this case there are the following possibilities:

(*a*)	$AB \parallel PQ$	$AC \parallel PR$	$BC \parallel QR$
(*b*)	$AB \nparallel PQ$	$AC \nparallel PR$	$BC \parallel QR$
(*c*)	$AB \nparallel PQ$	$AC \parallel PR$	$BC \nparallel QR$
(*d*)	$AB \parallel PQ$	$AC \nparallel PR$	$BC \nparallel QR$

No other case can occur, since, as was remarked earlier (section 2) and will be proved later [case (1) in section 11], if two pairs of the lines are parallel, the third pair must also be parallel.

In case (*a*) all three points are ideal points and therefore lie on the one ideal line. In case (*b*), as was remarked in section 2, $UV \parallel QR$. The point W in this case is an ideal point. Since $UV \parallel QR$, we see that the line UV passes through the point W; that is, the points U, V, and W are all on a line. Cases (*c*) and (*d*) are analogous to (*b*).

5. BASIC THEOREMS ABOUT PROJECTIVE LINES

THEOREM 1. *Through any two different points (ordinary or ideal) there passes one and only one projective line.*

There are three possibilities: (1) both points are ordinary points; (2) both points are ideal points; (3) one point is an ordinary point and the other an ideal point.

In case (1), we recall that according to the axioms of elementary geometry, through two different points there passes one and only one straight line. Since the ideal line does not contain any ordinary points, it cannot pass through either of the two given points. The theorem is proved for case (1).

In case (2), the points lie on the infinitely remote line of the

plane. Since each of the remaining lines contains only one ideal point, none of them can contain both the given points.

In case (3), we designate the ordinary point A and the ideal point B. The point B is determined (and can be represented on a drawing) by some line k. A projective line l will join the points A and B if and only if l is an ordinary line parallel to k and passing through the point A. As is known, such a line exists, and its uniqueness follows from the axioms concerning parallel lines.

THEOREM 2. *Two different projective lines intersect in one and only one point (ordinary or ideal).*

(Before reading the proof, try to prove this independently.)

In order to prove Theorem 2, we note first of all that there are two possibilities: (1) both lines are ordinary; (2) one line is ordinary and the other ideal. There cannot be two ideal lines, since we have agreed that there will be only one line in the plane which is infinitely remote.

In case (1), if the two ordinary lines are not parallel, then they have one ordinary point in common; if the given lines are parallel, then they have a unique ideal point in common.

In case (2), the ideal point of the ordinary line is the unique common point of the two given lines.

6. CENTRAL PROJECTION IN SPACE

Let there be given a plane π and a point S not on the plane (Fig. 8). We call the intersection point A' of the line SA with the plane π the *projection* on the plane π of the point A. Just as in the case of the plane, we call the point S the *center of projection.* We note that all the points of a plane τ which is parallel to the plane π and passes through the point S are mapped into ideal points of the plane π.

The plane π, that is, the plane in which the projections of the points lie, will be called *the plane of projection.*

THEOREM 3. *The projections of points on a line also lie on a line.*

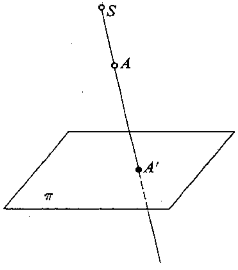

Fig. 8

For the proof, let us assume that we are given three points A, B, and C on a line a (Fig. 9). Then all the lines SA, SB, and SC lie in the same plane σ, determined by the point S and the line a.

If the planes σ and π are not parallel, the points A', B', and C', situated in both planes σ and π, all lie on one line, the line of intersection of these planes.

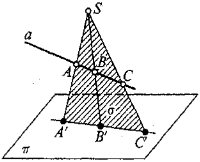

Fig. 9

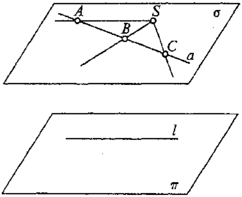

Fig. 10

In case the planes σ and π are parallel (Fig. 10), consider the ideal point of the line SA (considering the planes σ and π as *projective planes*). This ideal point must also belong to each line l which lies in the plane π and is parallel to SA. Therefore, in this case, the projection of the point A will be an ideal point A' of the plane π. The projections of the points B and C will also prove to be ideal points of the plane π. Since by agreement all the ideal points of the plane π lie on one line, the validity of Theorem 3 is established also in the case that σ and π are parallel.

2. The Theorem of Pappus and Pascal

Historical Note. Pappus of Alexandria was a Greek mathematician of the second half of the third century A.D. In his book, entitled *Mathematical Collection,* there are many fragments of Greek writings which have not come down to us in any other form. For this reason it is a valuable source of the history of ancient Greek mathematics.

Blaise Pascal (1623–1662) was an outstanding French mathematician, physicist, and philosopher. He received his first training in mathematics under the tutelage of his father—the celebrated mathematician Étienne Pascal. At the age of sixteen Blaise Pascal wrote his first scientific work. The Pappus-Pascal theorem, which is considered in this chapter, is a special case of one of the theorems proved in it. This special case, it is true, was already known to Pappus of Alexandria without the use of ideal points.

Pascal's mathematical interests were not limited to geometry. He designed and constructed an adding machine, wrote a series of works on arithmetic, algebra, the theory of numbers, and the theory of probability. In particular, he defined precisely the method of complete mathematical induction and used it for proofs. In physics Pascal studied barometric pressure and problems in hydrostatics. For example, he discovered the basic law of hydrostatics which states that pressure on the surface of a liquid produced by exterior forces is transmitted by the liquid equally in all directions.

7. THE PAPPUS-PASCAL THEOREM

We shall call an *n-gon* a figure formed by n different points of the plane, which will be numbered $1, 2, 3, \ldots, n$, and n lines which connect the points 1 and 2, 2 and 3, $\ldots$, $n - 1$ and n, n and 1. The points will be called the *vertices* of the n-gon, and the lines connecting consecutive vertices will be called its *sides*. It is clear that an n-gon defined in this way differs from the usual definition only in that the sides are taken to be *lines* and not *line segments.* If the points A, B, C, D, E, and F in that order are taken to be the vertices of a hexagon (see Fig. 11), then the vertices A and D, B and E, C and F, that is, the pairs which are separated by two vertices are said to

be *opposite* vertices. Two sides of a hexagon connecting vertices which are correspondingly opposite one another are said to be *opposite* sides of the hexagon. In the hexagon $ABCDEF$ considered above, the opposite sides are AB and DE, BC and EF, CD and FA.

The Pappus-Pascal theorem expresses a remarkable property of a hexagon when some special conditions are placed on its vertices.

THE PAPPUS-PASCAL THEOREM. *If the vertices of a hexagon lie alternately on two lines, then the points of intersection of its opposite sides all lie on a line.*

Even if all the vertices of the hexagon are ordinary points, this simple formulation of the theorem cannot be obtained without the introduction of ideal points and lines. Without these, in the conclusion of the theorem we would have to speak separately of the following three cases:

First case. Each pair of opposite sides intersect at an ordinary point, and the points of intersection (points P, Q, and R in Fig. 11) all lie on a line.

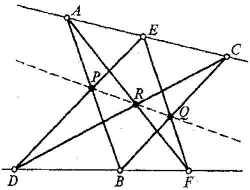

Fig. 11

Second case. The opposite sides of the hexagon are parallel (Fig. 12).

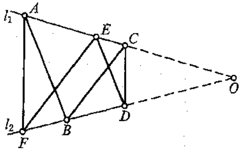

Fig. 12

Third case. Two pairs of opposite sides of the hexagon intersect in ordinary points P and Q, and the third pair of sides is parallel to the line PQ (Fig. 13).

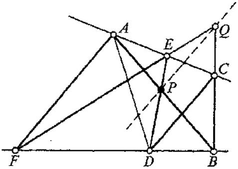

Fig. 13

Ideal elements enable us to include all three of these cases in the one formulation given above, since in the second case all three points of intersection of opposite sides are ideal points and therefore all lie on the ideal line of the given plane, and in the third case the ideal point of intersection of the third pair of opposite sides is on the line PQ, since parallel lines have one ideal point in common.

Furthermore, as will be shown, the Pappus-Pascal theorem is true even in case any of the vertices of the hexagon are ideal points. It goes without saying that for each of these cases a theorem could be formulated without the introduction of ideal points. But think how many cases there would be! And each of them would require a separate proof.

8. INTRODUCTION TO THE PROOF OF THE PAPPUS-PASCAL THEOREM

Before beginning to prove the Pappus-Pascal theorem, we shall prove an auxiliary proposition which will be needed in what follows.

LEMMA 1. *If we take a segment OC on the side OA of the angle AOB and a segment OD on the side OB such that* $\frac{OA}{OB} = \frac{OC}{OD}$, *then* $AB \parallel CD$ (Fig. 14).

Since the angle at O is common to both the triangles AOC and COD, and the sides adjacent to this angle are proportional, the triangles are similar. Therefore, the angles OCD and OAB are equal; hence, $CD \parallel AB$.

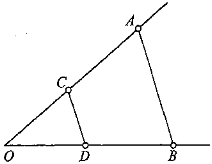

Fig. 14

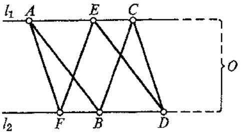

Fig. 15

Now we shall prove the *theorem of Pappus-Pascal for the second case*. Assume that $AB \parallel DE$ and $BC \parallel EF$. We need to show that $AF \parallel CD$. If the lines l_1 and l_2 on which the vertices of the hexagon are located are parallel (Fig. 15), then from our assumption it is clear that $BD = AE$ and $BF = CE$, since they are opposite sides of parallelograms. This means that the segments AC and DF are also equal. Therefore, the quadrilateral $ACDF$ is a parallelogram, and $AF \parallel CD$. If the lines l_1 and l_2 intersect in some point O (see Fig. 12), then, using the similarity of triangles, we get

$$\frac{OE}{OD} = \frac{OA}{OB} \quad \text{and} \quad \frac{OC}{OE} = \frac{OB}{OF}.$$

Termwise multiplication of these equalities gives $\frac{OC}{OD} = \frac{OA}{OF}$. From this last equality, in view of Lemma 1, it follows that $AF \parallel CD$. Thus, we have shown that if two pairs of opposite sides of a hexagon satisfying the conditions of the Pappus-Pascal theorem are parallel, then the third pair of sides is also parallel.

Let us prove another auxiliary proposition.

LEMMA 2. *Given an arbitrary line l in a plane σ, there exist a center of projection S and a plane of projection π such that the line l is projected onto the ideal line of the plane π. In this case, the plane π will be parallel to the plane τ which passes through the point S and the line l.*

For the proof choose the center of projection to be an arbitrary point S, not lying in the plane σ (Fig. 16). τ is the plane through the point S and the line l. (The possibility that the line l is ideal need not disturb us, for in this case, the plane through S and l is the plane through the point S parallel to the plane σ.) Let π be some plane

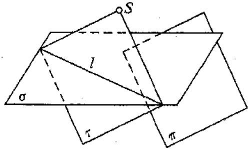

Fig. 16

parallel to τ, not passing through the point S. If a point A belongs to the line l, then the line SA lies in the plane τ and, therefore, is parallel to the plane π. Hence, the projection of A is an ideal point of the plane π. Since this is true for an arbitrary point on the line l, the point S and the plane π satisfy the conclusion of the lemma.

9. COMPLETION OF THE PROOF OF THE PAPPUS-PASCAL THEOREM

Let σ designate the plane in which the given hexagon $ABCDEF$ is located, and let l designate the line through PQ (Fig. 17). Accord-

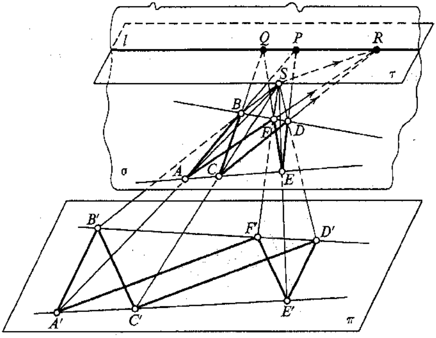

Fig. 17

ing to Lemma 2, there is a center of projection S and a plane of projection π such that the projection of the line l will be the ideal line of the plane π. Furthermore, the plane τ, determined by the point S and the line PQ, is parallel to the plane π. Let A', B', C', D', E', and F' be the projections of the vertices of the hexagon, and P', Q', and R' be the projections of the points P, Q, and R, respectively. Since the point P belongs to the lines AB and DE, it follows from Theorem 3 that the point P' belongs to the lines $A'B'$ and $D'E'$. By the same reasoning, we can show that Q' belongs to the lines $B'C'$ and $E'F'$, and R' to the lines $C'D'$ and $A'F'$. But P' and Q' are ideal points. Therefore, $A'B' \parallel D'E'$ and $B'C' \parallel E'F'$.

Furthermore, from Theorem 3 and the conditions of our theorem it follows that the points A', C', and E' lie on a line, and the points B', F', and D' also lie on a line. We have already proved (second case of the Pappus-Pascal theorem, section 8) that it follows from these conditions that $C'D' \parallel A'F'$. This means that the point R' lies on the ideal line of the plane π. But all points of the ideal line of this plane are, by construction, projections of points on the line l, which lies in the plane σ. Therefore, the point R belongs to the line l, that is, the points P, Q, and R lie on a line. Thus, the theorem is proved.

10. BRIANCHON'S THEOREM

It is easy to deduce the following consequence from the Pappus-Pascal theorem. It is sometimes called Brianchon's (1785–1864) theorem:

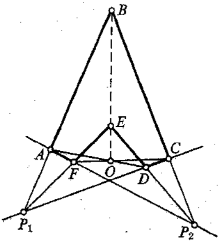

Fig. 18

THEOREM 4. *If the sides of a hexagon pass alternately through two given points, then the three lines connecting its opposite vertices pass through a common point.*

Let A, B, C, D, E, F be the successive vertices of a hexagon, and let the sides AB, CD, and EF pass through the point P_1 and the sides BC, DE, and AF pass through the point P_2 (Fig. 18). Let O be the point of intersection of the lines AD and CF. We need to prove that the line BE passes through the point O, or, in other words, that the points O, B, and E are all on a line. But this follows directly from the Pappus-Pascal theorem if it is applied to the hexagon ADP_2CFP_1, remembering that the sides of the hexagon are considered to be lines, and not line segments.

3. Desargues's Theorem

Historical Note. Gérard Desargues (1593–1662, or according to some authorities, 1591–1661) was a prominent French mathematician who laid the foundations of projective and descriptive geometry. Desargues was the first to introduce the discussion of ideal points and lines into geometry. Only the most outstanding mathematicians of his time were able to understand and evaluate his ideas during his lifetime. These were Descartes, Fermat, and Blaise Pascal. His ideas began to gain general recognition only at the beginning of the 19th century. A military engineer by education, Desargues was interested in the precise mathematical basis of practical operations. The works of Desargues concerning stonecutting and sundials were devoted to such questions.

11. DESARGUES'S THEOREM

Let us now go on to the second important configuration theorem, known as Desargues's theorem.

DESARGUES'S THEOREM. *If two triangles ABC and $A'B'C'$ are situated in the plane in such a manner that the lines AA', BB',* and *CC', connecting their corresponding vertices, all pass through one point O, then the points of intersection of corresponding*[1] *sides of these triangles all lie on one straight line* (Fig. 19).

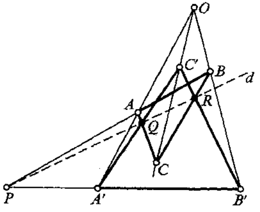

Fig. 19

[1] The vertices A and A', B and B', C and C' are considered corresponding by definition. Sides are said to correspond if they connect corresponding vertices.

In passing, we note that the theorem formulated in section 2 of the Introduction is a special case of Desargues's theorem, applied to the triangles ABC and PQR.

We proceed now to the proof of Desargues's theorem. Let AB and $A'B'$ intersect in the point P, AC and $A'C'$ in the point Q, and BC and $B'C'$ in the point R. We need to prove that the points P, Q, and R all lie on a line.

We consider two cases: (1) at least two points are ideal points, and (2) at least two points are ordinary points.

For case (1), we assume that among the points P, Q, and R, two, say P and Q, are ideal points. Then $AB \parallel A'B'$ and $AC \parallel A'C'$. If the point O is an ordinary point (Fig. 20), then from the similarity of the triangles AOB and $A'OB'$ it follows that $\frac{OB}{OB'} = \frac{OA}{OA'}$, and from the similarity of the triangles AOC and $A'OC'$ it follows that $\frac{OA}{OA'} = \frac{OC}{OC'}$. Hence, $\frac{OB}{OB'} = \frac{OC}{OC'}$. From Lemma 1, proved in section 8, it follows from the last equality that $BC \parallel B'C'$; that is, the point R is also an ideal point.

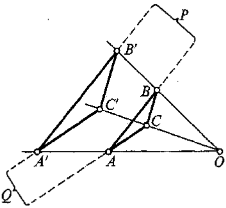

Fig. 20

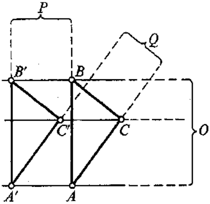

Fig. 21

If the point O is an ideal point (Fig. 21), then we have $AA' \parallel BB' \parallel CC'$. Thus, the quadrilaterals $AA'B'B$ and $AA'C'C$ are parallelograms. Hence $BB' = AA'$ and $AA' = CC'$. Consequently, the segments BB' and CC' are equal and parallel, which implies that $BC \parallel B'C'$ and thus the point R is an ideal point. Thus, the points P, Q, and R all lie on an ideal line, and Desargues's theorem is now proved in case P and Q are ideal points.

For case (2), we assume that among the points P, Q, and R, two, for example, the points P and Q, are ordinary (Fig. 22). Let σ

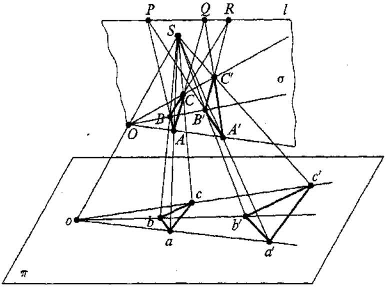

Fig. 22

designate the plane in which the given triangles are situated, and let l be the line PQ. According to Lemma 2, proved in section 8, there is a center of projection S and a plane of projection π such that the projection of the line l serves as the ideal line of the plane π. We shall designate the projections of points denoted by capital letters by the corresponding small letters. Since the point P belongs to the lines AB and $A'B'$, from Theorem 3 (section 6) it follows that the point p belongs to the lines ab and $a'b'$. By the same reasoning, it can be shown that q belongs to the lines ac and $a'c'$, r to the lines bc and $b'c'$, and o to the lines aa', bb', and cc'. Furthermore, since p and q are ideal points, we see that $ab \parallel a'b'$ and $ac \parallel a'c'$. We have already proved in case (1) that this implies that $bc \parallel b'c'$. This means that the point r lies on the ideal line of the plane π. But, by construction, all the points of this ideal line are projections of the points of the line l. Consequently, the point R belongs to the line l, that is, the points P, Q, and R all lie on the same line.

The other possibilities for the points P, Q, and R differ from these only in notation, since these points enjoy completely equivalent properties. Thus, we can say that Desargues's theorem is completely proved.

12. ALTERNATIVE PROOFS OF DESARGUES'S THEOREM

It is interesting to note that the proof of Desargues's theorem can be carried out without the use of the theory of similar triangles. But then solid geometry must be used to a much greater extent. Let us carry out this proof.

(*a*) First, we prove a lemma.

LEMMA 3. *Let α, β, and γ be three different planes. Let h be the line of intersection of planes α and β, k the line of intersection of planes α and γ, and l the line of intersection of planes β and γ. If the lines h, k, and l are all different, then they have exactly one point in common.*

For the proof we consider the lines h and k. These cases are possible: (1) all three lines are ordinary lines and $h \parallel k$ (Fig. 23); (2) all three lines are ordinary and h and k have an ordinary point U in common (Fig. 24); (3) one of the three lines is ideal (h in Fig. 25). (No more than one of the lines h, k, and l can be ideal, since a plane contains only one ideal line.)

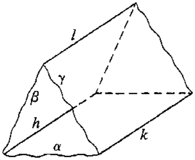

Fig. 23

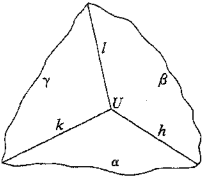

Fig. 24

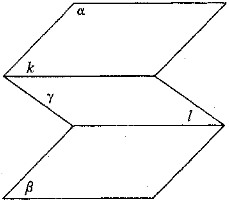

Fig. 25

For case (1), we use a theorem proved in a school geometry course: If a given line is parallel to some other line in some plane, then the given line is parallel to this plane. Thus, the line k must be parallel to the plane β (Fig. 23). Thus, the plane γ passes through the line k, which is parallel to the plane β. As is known from a school course, it follows from this that the line of intersection of the planes β and γ, that is, the line l, is parallel to the line k. Thus, $h \parallel k \parallel l$, and, consequently, these lines have the one ideal point U in common.

In case (2), we note that since the point U lies on the line h, this point belongs to the planes α and β. But the point U also lies on the line k and, therefore, belongs to the planes α and γ. This means that the point U is a point common to the planes β and γ and so must belong to their line of intersection—the line l. Thus, our lemma is proved for the second case.

In case (3), we assume that h is an ideal line. (If some other of the three lines h, k, and l is ideal, we reason analogously.) Then $\alpha \parallel \beta$ (Fig. 25). According to our condition, both k and l are ordinary lines; that is, γ intersects both plane α and plane β. From a school course it is known that inasmuch as $\alpha \parallel \beta$, these lines of intersection must be parallel, that is, $k \parallel l$. The common ideal point U of the lines k and l, of course, lies on the ideal line of the plane α, that is, on the line h. Thus, our three lines have the one point U in common in the third case, as well. Lemma 3 is proved completely.

(*b*) We now give a new proof of Desargues's theorem. It is only necessary to consider the case when two of the points P, Q, and R are ideal. As has already been shown, the other cases of Desargues's theorem can be deduced from this one without recourse to the theory of similar triangles. Assume that P and Q are ideal points. We wish to prove that R is also an ideal point.

To begin with, assume that O is an ordinary point. Let σ designate the plane in which the given triangles are situated (Fig. 26). Let τ be a plane parallel to the plane σ and different from σ. Further, let a be a line through the point A, not belonging to the plane σ. Let the lines b, c, and o be parallel to a, such that b passes through B, c through C, and o through O. Let the lines a, b, c, and o intersect the plane τ in the points $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{O}$. The lines AB and $\bar{A}\bar{B}$ are lines of intersection of the plane passing through the lines a and b with the parallel planes σ and τ. Therefore,

$$AB \parallel \bar{A}\bar{B}.$$

Similarly,

$$AC \parallel \bar{A}\bar{C} \quad \text{and} \quad BC \parallel \bar{B}\bar{C}.$$

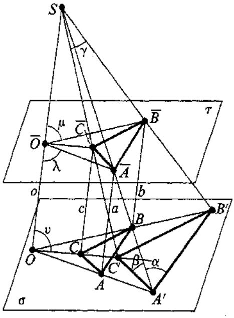

Fig. 26

We also have the following planes:

λ, passing through lines a and o and points $O, A, \bar{O}, \bar{A}, A'$,
μ, passing through lines b and o and points $O, B, \bar{O}, \bar{B}, B'$,
ν, passing through lines c and o and points $O, C, \bar{O}, \bar{C}, C'$.

Since P and Q are ideal points by assumption,

$$AB \parallel A'B' \quad \text{and} \quad AC \parallel A'C'.$$

From $AB \parallel \bar{A}\bar{B}$ it follows that

$$A'B' \parallel \bar{A}\bar{B},$$

and from $AC \parallel \bar{A}\bar{C}$ it follows that

$$A'C' \parallel \bar{A}\bar{C}.$$

Therefore, we can also consider the planes

α, passing through the points $A', B', \bar{A}$, and $\bar{B}$,
β, passing through the points $A', C', \bar{A}$, and $\bar{C}$.

The planes α, λ, μ and the lines $A'\bar{A}$, $B'\bar{B}$, and $O\bar{O}$ satisfy the conditions of Lemma 3. Therefore, the lines $A'\bar{A}$, $B'\bar{B}$, and $O\bar{O}$ have a point S in common. We can also apply Lemma 3 to the planes β, ν, λ and the lines $A'\bar{A}$, $C'\bar{C}$, $O\bar{O}$. This means that these lines also intersect in a common point. But, as we have shown, S is the point of intersection of the lines $A'A$ and OO. Therefore, the line $C'\bar{C}$ also passes through the point S. We now consider the plane

γ, passing through the points S, B', and C'.

It is clear that the points $\bar{B}$ and $\bar{C}$ are situated in this plane. Therefore, the plane γ intersects the plane σ in the line $B'C'$, and the plane τ in the line $\bar{B}\bar{C}$. Since $\sigma \parallel \tau$, $B'C' \parallel \bar{B}\bar{C}$. But it was established above that $\bar{B}\bar{C} \parallel BC$. Thus $B'C' \parallel BC$ and R is an ideal point, which was to be proved.

Now we assume that the point O is an ideal point (Fig. 27); hence $AA' \parallel BB' \parallel CC'$. It is necessary to change the proof somewhat, since it is not possible in this case to draw the line o through O parallel to the line a. Suppose, as before, that P and Q are ideal points; hence $AB \parallel A'B'$ and $AC \parallel A'C'$. To prove that R is an ideal point, it is necessary to prove that $BC \parallel B'C'$.

Again let σ denote the plane in which the given triangles are situated, and let τ denote any arbitrary plane parallel to the plane σ and different from it. As before, let a, b, and c be

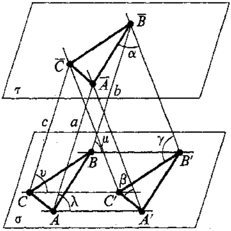

Fig. 27

three parallel lines drawn through the points A, B, and C, respectively, intersecting the plane τ in the points $\bar{A}$, $\bar{B}$, and $\bar{C}$. As before, the corresponding sides of the triangles ABC and $\bar{A}\bar{B}\bar{C}$ are parallel; also $\bar{A}\bar{B} \parallel A'B'$ and $\bar{A}\bar{C} \parallel A'C'$.

Let λ designate the plane $A\bar{A}A'$, μ the plane $B\bar{B}B'$, and ν the plane $C\bar{C}C'$. The planes α, β, and γ will be defined as before. It is clear that the plane α intersects the planes λ and μ in the parallel lines $A'\bar{A}$ and $B'\bar{B}$, while the line of intersection of the planes λ and μ is an ideal line, which we shall designate p.[1] Thus, the planes α, λ, and μ satisfy the conditions of the third case of Lemma 3. Consequently, the lines $A'\bar{A}$, $B'\bar{B}$, and p have a common (ideal) point S. Applying Lemma 3 to the planes β, λ, and ν, exactly in the same way we establish that the lines $A'\bar{A}$, $C'\bar{C}$, and p have a point S' in common. But since the lines $A'\bar{A}$ and p have only one common point (see Theorem 2 of section 5), the points S and S' must coincide. The points S, B', and C' define the plane γ, in which the points $\bar{B}$ and $\bar{C}$ also lie. Thus, the lines $B'C'$ and $\bar{B}\bar{C}$ must be parallel, since these are the lines of intersection of the plane γ with the parallel planes σ and τ. But since $BC \parallel \bar{B}\bar{C}$, we have $B'C' \parallel BC$, and R is an ideal point, which was to be proved.

(*c*) Let us also prove that Desargues's theorem can be deduced from the Pappus-Pascal theorem without explicitly using the theory of similar triangles. (Recall, however, that the theory of similar triangles was used in the proof of the Pappus-Pascal theorem.) As above, it suffices to consider the case when P and Q are ideal points.

Let (Fig. 28) it be given that the three lines AA', BB', and CC' intersect at point O and that $AB \parallel A'B'$ and $AC \parallel A'C'$. It is necessary to prove that $BC \parallel B'C'$. Let us draw the line through A parallel to OB, and let it intersect the line $A'C'$ at the point L, and the line OC at the point M. Let the letter N designate the intersection of the line AB with the line $B'L$ and join the point N with the points O and M. Applying the theorem of Pappus-Pascal to the hexagon $ONALA'B'$ and using the fact that, by assumption, $AN \parallel A'B'$, and, by construction, $AL \parallel B'O$, we find that $ON \parallel LA'$ and, therefore, $ON \parallel CA$. In the hexagon $ONMACB$ we have from the above that $ON \parallel AC$ and, by construction, that $AM \parallel OB$. Therefore, by the theorem of Pappus-Pascal, we must have $MN \parallel BC$. Finally, we note that the hexagon $ONMLC'B'$ also satisfies the condition of the theorem of Pappus-Pascal; $ON \parallel LC'$ as was proved, and $ML \parallel B'O$ by construction. Therefore, $MN \parallel B'C'$. But we have already established that $MN \parallel BC$. Therefore, $BC \parallel B'C'$, which was to be proved.

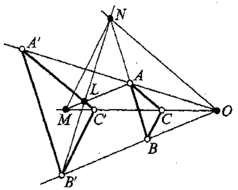

Fig. 28

[1] Since the planes μ, ν, and λ are all parallel, the ideal line p is the common intersection of all three of these planes.

Remark. We note that it has been shown to be impossible to prove Desargues's theorem unless we use the theory of similar triangles or unless we employ constructions in space. This can be rigorously proved.

13. THE CONVERSE OF DESARGUES'S THEOREM

THEOREM 5. *If two triangles ABC and $A'B'C'$ are so situated in a plane that the points of intersection of their corresponding sides all lie on one line, then the lines AA', BB', and CC' connecting corresponding vertices of these triangles intersect in a point.*

For the proof consider Fig. 19. Let it be known that P, Q, and R all lie on one line. We are to prove that the line CC' passes through the point of intersection O of the lines AA' and BB'. We shall apply Desargues's theorem to the triangles $AA'Q$ and $BB'R$, letting A and B, A' and B', Q and R be the corresponding vertices. The lines connecting the corresponding vertices of these triangles intersect at the one point P. Therefore, by Desargues's theorem, the points of intersection of their corresponding sides AA' and BB', AQ and BR, $A'Q$ and $B'R$, that is, the points O, C, and C', must all lie on one line. Thus, the theorem is proved.

The well-known theorem about the intersection of the medians of a triangle at a point is an immediate consequence of Theorem 5. In fact, if L, M, and N are the mid-points of the sides of the triangle ABC (Fig. 29), then the corresponding sides of the triangles ABC and MNL are parallel as a consequence of the theorem concerning the mid-points of the sides of a triangle; that is, they intersect on the one ideal line of the plane. From this, according to Theorem 5, it follows that the lines AM, BL, and CN, that is, the medians of the given triangle, must intersect in one point.

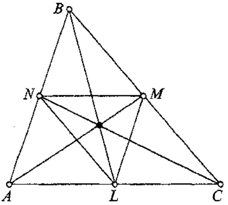

Fig. 29

4. Some Properties of Polygons

14. SOME PROPERTIES OF QUADRILATERALS

In this section we shall establish several properties of quadrilaterals with the help of the theorem of Desargues.

THEOREM 6. *Let $ABCD$ (Fig. 30) be an arbitrary quadrilateral, E the point of intersection of the opposite sides AB and CD, F the point of intersection of the diagonals AC and BD, M the point of intersection of the line EF with the side AD. Then the point P, at which the lines AB and CM intersect, the point Q at which the lines BM and CD intersect, and the point R at which the sides AD and BC intersect, all lie on one line.*

For the proof, consider the triangles AED and CMB. The lines AC, EM, and BD intersect at the point F (see Table 1). Corresponding sides of these triangles intersect at the points P, Q, and R. According to Desargues's theorem, these points are situated on one line, which was to be proved.

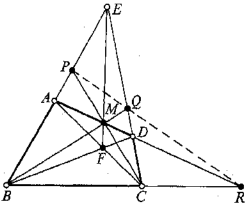

Fig. 30

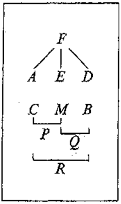

Table 1

COROLLARY. *The lines connecting the ends of a base of a trapezoid or a parallelogram with the mid-point of the opposite base intersect the lateral sides at points on a line which is parallel to the base.*

Indeed, if $ABCD$ is a trapezoid with bases AD and BC (Fig. 31), then, as is known, the point M, constructed as in Theo-

rem 6, is the mid-point of the segment AD.[1] On the other hand, the point of intersection R of the lines AD and BC is in this case infinitely remote, so that the line PQ, which must pass through this point, will be parallel to the lines AD and BC.

If $ABCD$ is not a trapezoid, but a parallelogram, then the argument is the same except that in this case E is an ideal point.

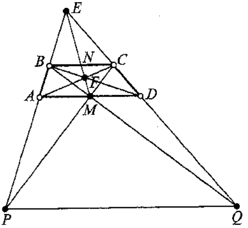

Fig. 31

15. SOME PROPERTIES OF PENTAGONS

In this section we shall establish several properties of pentagons with the help of the Pappus-Pascal theorem and the theorem of Desargues.

THEOREM 7. *Let ABCDE be an arbitrary pentagon, F the point of intersection of the nonadjacent sides AB and CD, M the point of intersection of the diagonal AD with the line EF. Then the point of intersection P of the side AE with the line BM, the point of intersection Q of the side DE with the line CM, and the point of intersection R of the side BC with the diagonal AD all lie on one line* (Fig. 32).

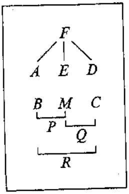

Table 2

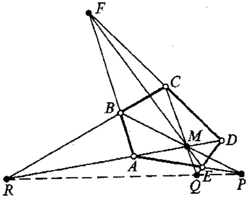

Fig. 32

[1] Here is a sketch of the proof of this statement. Since $\triangle AME$ is similar to $\triangle BNE$ and $\triangle MDE$ is similar to $\triangle NCE$, we find that $\frac{AM}{MD} = \frac{BN}{NC}$ (Fig. 31). Since $\triangle AMF$ is similar to $\triangle NCF$ and $\triangle MDF$ is similar to $\triangle BNF$, we find that $\frac{MD}{AM} = \frac{BN}{NC}$. The two equalities show that $AM = MD$.

For the proof, it suffices to note that the triangles AED and BMC satisfy the conditions of Desargues's theorem, since the lines AB, EM, and DC intersect at the point F. From this it follows directly that the points P, Q, and R all lie on one line (see Table 2).

THEOREM 8. *Let $ABCDE$ be an arbitrary pentagon, F the point of intersection of the two nonadjacent sides AB and CD, N the point of intersection of two other nonadjacent sides AE and BC, H the point of intersection of the diagonals AD and CE, K the point of intersection of the lines EF and BH. Then the line DK passes through the point N* (Fig. 33).

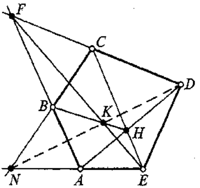

Fig. 33

For the proof of this theorem it suffices to note that the hexagon $AEFCBH$ satisfies the conditions of the Pappus-Pascal theorem with the sides AE and BC intersecting at the point N, the sides EF and BH at the point K, the sides FC and HA at the point D. Therefore, the points N, K, and D all lie on one line; that is, the line DK passes through the point N. Thus, the theorem is proved.

16. MORE PROPERTIES OF QUADRILATERALS

THEOREM 9. *Given a trapezoid (or parallelogram) $ABCD$* (see Fig. 34 and Fig. 35) *and four lines, BQ, CQ, AR, DR, with $BQ \parallel AR$, $CQ \parallel DR$, then P, the point of intersection of the lateral sides, lies on the line QR.*

For the proof it suffices to show that the points P, Q and R all lie on the same line. The sides of the triangles BCQ and ADR are correspondingly parallel, that is, the points of intersection of their corresponding sides all lie on the same ideal line. According to

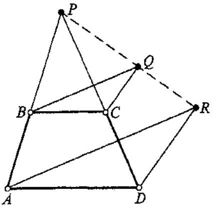

Fig. 34

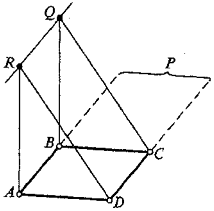

Fig. 35

Theorem 5, the lines AB, DC, and RQ must therefore intersect in a common point. Thus, the line QR passes through the point P, and the theorem is proved. (We note that if $ABCD$ is a parallelogram, then the point P is an ideal point, so that $QR \parallel AB$ as in Fig. 35.)

We shall give an example of a still more complicated configuration theorem.

THEOREM 10. *Let P and Q be the points of intersection of opposite sides of the quadrilateral $ABCD$* (Fig. 36), *S the point of intersection of the lines AC and PQ, T the point of intersection of the lines BD and PQ. Further let the quadrilateral $KLMN$ be such that its opposite sides intersect in the points P and Q, and such that the diagonal KM passes through the point S. Then the diagonal LN passes through the point T.*

For the proof draw the lines AK, DL, CM, and BN. The triangles ACD and KML satisfy the conditions of Theorem 5 (see Table 3).

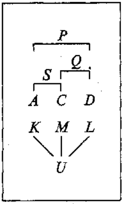

Table 3

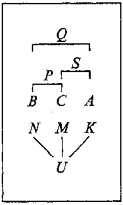

Table 4

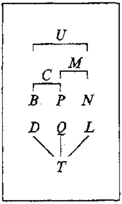

Table 5

Therefore, the lines AK, CM, and DL pass through one point U. Theorem 5 can also be applied to the triangles BCA and NMK (Table 4). Therefore, the lines BN, CM, and AK also intersect in the

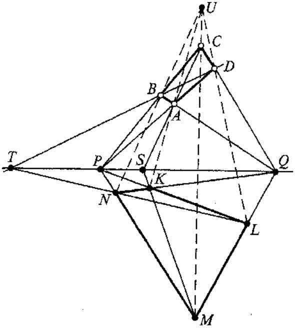

Fig. 36

point U. Now Theorem 5 can be applied to the triangles BPN and DQL (Table 5). Therefore, the lines BD, PQ, and NL intersect at the point T. Thus, Theorem 10 is proved.

5. Problems

17. INACCESSIBLE POINTS OR LINES

Configuration theorems are especially useful for solving problems of construction that involve so-called "inaccessible" points or lines. These often occur in practical problems.

For example, if on a drawing board a draftsman has lines a and b which intersect beyond the limits of the board (Fig. 37), then although the point of intersection exists, no drafting instruments can be applied to it. Also, in the practice of surveying, some point may be inaccessible to the measuring instrument by being situated in a marshy spot or above the surface of the earth.

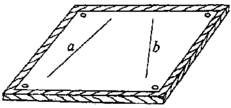

Fig. 37

Similarly, it may sometimes be impossible to draw a line between two points. In drafting, it may happen that a line must cross a region where there is an instrument which cannot be moved (a pantograph, a planimeter). In surveying, it often happens that some obstacle makes it impossible to place stakes along some line or to place a measuring tape on it.

Various cases of inaccessible points and lines which occur in practice can be described mathematically as follows:

We shall call a given point an *inaccessible point* if it is determined by two given intersecting lines, in such a manner that in the course of the given problem the point of intersection cannot be used for construction. An inaccessible point P defined by the lines a and b (Fig. 38) we shall designate $P(a,b)$.

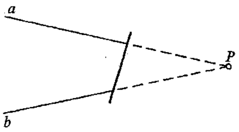

Fig. 38

We shall call a given line an *inaccessible line* if it is determined by two given points (whether accessible or inaccessible), in such a manner that this line cannot be used for construction. An inaccessible line p defined by the points A and B we shall designate $p(A,B)$.

18. CONSTRUCTIONS INVOLVING INACCESSIBLE POINTS OR LINES

We shall give several examples of the solution of problems with inaccessible elements using configuration theorems. Also, configuration theorems often enable us to solve problems of construction by using only a straightedge.

PROBLEM. 1. Given an ordinary point Q and an inaccessible point $P(a,b)$, draw the line PQ.

FIRST SOLUTION (using the Pappus-Pascal theorem). We first choose arbitrary points A and B on the line a, and arbitrary points D and E on the line b (Fig. 39). Draw AE and BD, BQ and EQ. Let C be the point of intersection of the lines AE and BQ; we shall agree to write this as $C \equiv AE \times BQ$. Further (using the notation just shown), let $F \equiv BD \times EQ$. Draw AF and CD and let $R \equiv AF \times CD$. From the theorem of Pappus-Pascal applied to the hexagon $ABCDEF$, it follows that the line QR passes through the point P, that is, coincides with the line PQ. Therefore, in order to solve the problem, it suffices to connect the points Q and R by means of a straightedge.

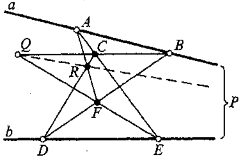

Fig. 39

SECOND SOLUTION (using Desargues's theorem). Let O (Fig. 40) be an arbitrary point and p_1, p_2, and p_3 three arbitrary lines passing through the point O.

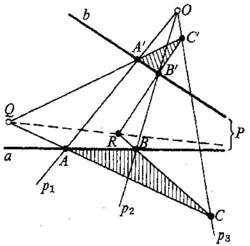

Fig. 40

Let us now consider the intersections of these lines. Let

$$a \times p_1 \equiv A, \qquad b \times p_1 \equiv A',$$
$$a \times p_2 \equiv B, \qquad b \times p_2 \equiv B',$$
$$AQ \times p_3 \equiv C, \qquad A'Q \times p_3 \equiv C',$$
$$BC \times B'C' \equiv R.$$

Considering the triangles ABC and $A'B'C'$, we note that according to Desargues's theorem, the points P, Q, and R all lie on one line. Therefore, the line we are looking for coincides with the line QR.

PROBLEM 2. Given an ordinary line q and an inaccessible line $p(A,B)$, construct the point of intersection C of these lines.

It is convenient to solve this problem with the use of Brianchon's theorem (section 10). Draw (Fig. 41) through the point A two arbi-

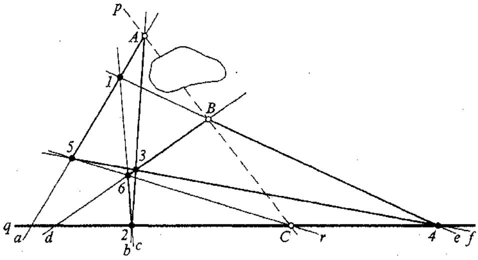

Fig. 41

trary lines a and b, and through the point B two arbitrary lines d and e. Let c denote the line connecting the point $1 \equiv a \times e$ with the point $2 \equiv b \times q$, f the line connecting the points $3 \equiv b \times d$ and $4 \equiv e \times q$, and r the line connecting the points $5 \equiv a \times f$ and $6 \equiv c \times d$.

Considering the hexagon $A54B62$, we note that its sides $A5$, $4B$, and 62 pass through the point 1, and its sides 54, $B6$, and $2A$ pass through the point 3. Therefore, according to Brianchon's theorem, the lines AB, 56, and 42, that is, the lines AB, r, and q, must meet

in a point. Therefore, the point of intersection C of the line q with the inaccessible line AB can be constructed as the intersection of the lines q and r.

PROBLEM 3. Construct the point of intersection M of two inaccessible lines AA' and BB'.

Let P (Fig. 42) be the point of intersection of lines AB and $A'B'$. Let d be an arbitrary line through this point and choose two arbitrary

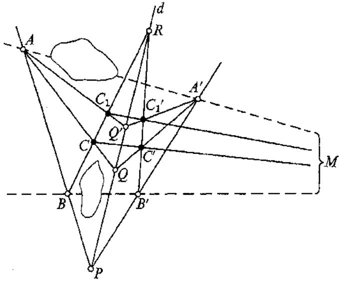

Fig. 42

points Q and R (different from P) on this line. If $C \equiv AQ \times BR$ and $C' \equiv A'Q \times B'R$, then the line CC' passes through the point M as a direct consequence of Theorem 5 of section 13, applied to the triangles ABC and $A'B'C'$.

Changing the position of the line d or the position of either of the points Q and R, we can construct in the same manner another pair of points C_1 and C_1' such that the line through them also goes through the point M. Then we can construct the desired point M as the point of intersection of the lines CC' and C_1C_1'.

PROBLEM 4. Construct a line RQ if $R(a,a')$ and $Q(b,b')$ are two inaccessible points.

Let C' (Fig. 43) be the point of intersection of the lines a' and b' and C the point of intersection of the lines a and b. On the line CC' choose an arbitrary point O (different from C and C'). Through

the point O draw two arbitrary lines p and q (different from the line CC'). Let B and B' be the points of intersection of the line p with the lines b and b' respectively, A and A' the points of intersection of the line q with the lines a and a' respectively. Applying Desargues's theorem to the triangles ABC and $A'B'C'$, we see immediately that the point $P_1 \equiv AB \times A'B'$ is on the line QR.

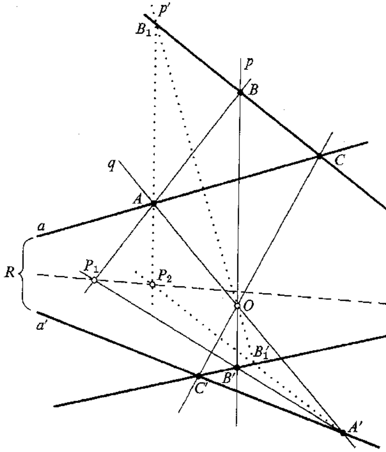

Fig. 43

In the same way, shifting the position of the point O on the line CC' or changing the lines p and q, we can find another point P_2 on the line QR. (In Fig. 43, p has been changed to p'.) Then the line P_1P_2 is the line we are looking for.

Problem 5. Suppose we are given two parallel lines a and b and a point Q. Using only a straightedge, draw a line through the point Q parallel to the given lines.

Let c (Fig. 44) be any line passing through Q and intersecting

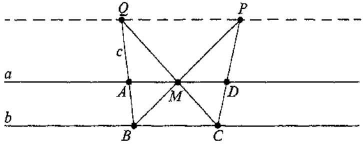

Fig. 44

the given lines a and b in the points A and B, respectively. Let D be any arbitrary point (different from A) on line a. Now construct any trapezoid with nonparallel lateral sides using AD as one of its bases and locating the other base at any place on line b; then construct the point M as in Theorem 6 of section 14 (construction lines not shown). As remarked in the corollary to that theorem, M will be the mid-point of the segment AD. Now let $C \equiv QM \times b$ and $P \equiv CD \times BM$. By the corollary to Theorem 6, PQ is the desired line.

19. PROBLEMS FOR SOLUTION BY THE READER

PROBLEM 6. Through a given inaccessible point draw a line parallel to two given parallel lines.

PROBLEM 7. Through a given point draw a line parallel to two given parallel lines, both of which are inaccessible.

PROBLEM 8. Prove: If the sides and diagonals of one quadrilateral are parallel to the sides and diagonals of another quadrilateral, then these quadrilaterals are homothetic[1] (similarly placed).

PROBLEM 9. Prove: If $ABCD$ and $AB'CD'$ are two parallelograms, then the lines BB' and DD' are parallel and the lines BD' and $B'D$ are parallel.

PROBLEM 10. Determine which of the preceding problems can be used to solve the following problems:

[1] The two quadrilaterals $ABCD$ and $A'B'C'D'$ are said to be homothetic if there is a point O such that $OA/OA' = OB/OB' = OC/OC' = OD/OD'$.

1) Two lines are marked on the sea with buoys. While remaining on an island, from which the buoys can be seen, it is necessary to mark with stakes the line going from a given point A on the island to the point of intersection of the lines determined by the buoys (Fig. 45).

2) How can you find the point of intersection of two lines a and b if there is an elevation on the line a making it impossible to sight along it (Fig. 46)?

3) Solve the same problem in the case where there are obstructions on both of the lines (Fig. 47).

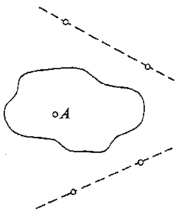

Fig. 45

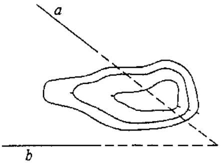

Fig. 46

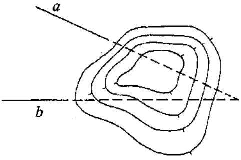

Fig. 47

4) Between two electric transmission towers A and B (Fig. 48) and in line with them there is to be placed a third tower C. How

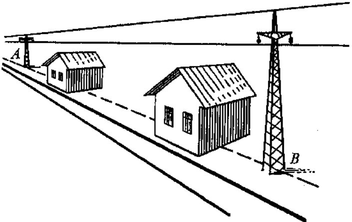

Fig. 48

should it be located if there are two structures situated between towers A and B?

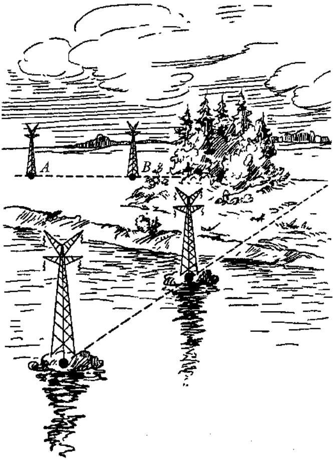

Fig. 49

5) Find a way of determining, without setting foot off dry land, the point of intersection of the two electric power lines under construction which are shown in Fig. 49.

6. The Algebraic Meaning of Configuration Theorems

20. ALGEBRAIC IDENTITIES AS CONFIGURATION THEOREMS

We have encountered several important configuration theorems and have acquired some idea about their possible applications. A number of questions naturally arise: Are there more configuration theorems? Are there configuration theorems other than those which follow from Desargues's theorem and the Pappus-Pascal theorem? Isn't there some general method for discovering the configurative properties of the plane? These questions lead us into a new branch of research. Many of its results have been developed only in the last two decades. There are still a number of questions which remain to be answered. We shall give here just a fleeting suggestion of this part of mathematics.

As early as the 17th century, while searching for a general method of geometric investigation, mathematicians had arrived at the idea of coordinates, making possible the application of algebraic and computational methods to geometry. At the beginning of the 20th century, the important role of the theorems of Pappus-Pascal and Desargues, connecting geometry with algebra and arithmetic, was discovered. Finally, the investigations of the last fifteen years have established that every configuration theorem can be "translated into algebraic language" as some algebraic identity, and, conversely, every algebraic identity can be stated geometrically as some configuration theorem.

Let us consider the simplest of examples.

Let two arbitrary numbers a and b be given. Choosing some segment, as OE in Fig. 50, as the unit of length, let us represent these numbers as line segments. Then the product ab of the given numbers can also be represented as a line segment. This segment can be

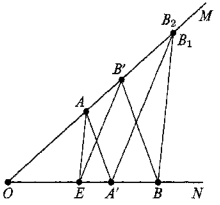

Fig. 50

constructed as follows: Take an arbitrary angle MON. On side OM lay off the *first* factor of ab, that is, the segment $OA = a$, and on the other side, ON, the *second* factor, $OB = b$, and the unit, $OE = 1$. Construct the line $BB_1 \parallel AE$. Then

$$\frac{OB_1}{OB} = \frac{OA}{OE} \quad \text{or} \quad \frac{OB_1}{b} = \frac{a}{1}; \quad \text{hence,} \quad OB_1 = ab.$$

By the same method, construct the product ba. For this lay off on side OM the *first* factor of ba, that is, the segment $OB' = b$, and on the side ON the *second* factor, $OA' = a$. Construct $A'B_2 \parallel B'E$. Then

$$\frac{OB_2}{OA'} = \frac{OB'}{OE} \quad \text{or} \quad \frac{OB_2}{a} = \frac{b}{1}; \quad \text{hence,} \quad OB_2 = ba.$$

By our first construction we have $AE \parallel BB_1$. Since $OA = OA' = a$ and $OB' = OB = b$, it follows from Lemma 1 of section 8 that $AA' \parallel BB'$. Hence, by applying the Pappus-Pascal theorem to the hexagon $AA'B_1BB'E$, we find $A'B_1 \parallel B'E$. However, by our second construction we have $A'B_2 \parallel B'E$. This means that if the Pappus-Pascal theorem is true, then the point B_2 coincides with the point B_1, that is, $OB_1 = OB_2$ or, what is the same, $ab = ba$. Thus, "in algebraic language" the Pappus-Pascal theorem becomes the commutative law of multiplication.

21. SCHEMATIC NOTATION FOR CONFIGURATION THEOREMS

After deeper study of the problem and upon acquiring the necessary experience, a person can learn to state various algebraic

identities as configuration theorems. Since one can construct as many identities as one likes, one can also construct as many configuration theorems as one wants. Furthermore, the number of points and lines involved in a theorem can be made arbitrarily large.

The more complicated a configuration theorem is, the more difficult becomes its verbal formulation. For this reason, we have adopted a scheme of notation for complicated configuration theorems. For example, let us state the Pappus-Pascal theorem using this scheme. According to the condition of this theorem (section 7) the points A, C, and E all lie on one line. In order to express this condition schematically, the letters representing the points are written in a row:

$$ACE$$

The same condition is imposed on the points B, D, and F, and, therefore, another row is added to the scheme:

$$BDF$$

Further, the point P appears at the point of intersection of the lines AB and DE. This construction is indicated schematically thus:

$$\left.\begin{matrix} AB \\ DE \end{matrix}\right\} P$$

The construction of the points Q and R is indicated in the same manner. The complete schematic notation of the Pappus-Pascal theorem takes the following form:

$$\begin{array}{l} ACE \\ BDF \\ \left.\begin{matrix} AB \\ DE \end{matrix}\right\} P \\ \left.\begin{matrix} BC \\ EF \end{matrix}\right\} Q \\ \left.\begin{matrix} CD \\ AF \end{matrix}\right\} R \\ \hline \quad PQR \end{array}$$

The last notation under the line expresses the conclusion of the theorem: "The points P, Q, and R lie on one line."

Here is an example of the schematic notation of a more complicated configuration theorem which deals with 14 points (the points are designated by numerals) and 16 lines (the lines arise in construction):

$$\begin{array}{l}
\left.\begin{array}{rr} 1 & 2 \\ 3 & 5 \end{array}\right\} 7 \\
\left.\begin{array}{rr} 2 & 4 \\ 3 & 6 \end{array}\right\} 8 \\
\left.\begin{array}{rr} 1 & 3 \\ 4 & 7 \end{array}\right\} 9 \\
\left.\begin{array}{rr} 2 & 9 \\ 7 & 8 \end{array}\right\} 10 \\
\left.\begin{array}{rr} 1 & 2 \\ 3 & 10 \end{array}\right\} 11 \\
\left.\begin{array}{rr} 2 & 6 \\ 3 & 4 \end{array}\right\} 12 \\
\left.\begin{array}{rr} 6 & 7 \\ 5 & 11 \end{array}\right\} 13 \\
\left.\begin{array}{rr} 1 & 5 \\ 7 & 12 \end{array}\right\} 14 \\
\hline
2 \quad 13 \quad\;\; 14
\end{array}$$

This theorem is represented in Fig. 51. We can construct the whole figure by first choosing the points *1*, *2*, *3*, *4*, *5*, and *6* arbitrarily. The positions of all the remaining points will be then completely determined. It would be quite complicated for us to derive the algebraic identity into which this configuration can be translated, and we shall just state that the identity is

$$a\,(bc) + d = (ab)\,c + d.$$

It would also be lengthy to prove this configuration theorem by geometric arguments using the theorems of Desargues and Pappus-Pascal.

Finally, it has been shown that all configuration theorems, no matter how complicated, can be deduced from the Pappus-Pascal theorem. Remember that in section 12(*c*) we did deduce Desargues's theorem from the Pappus-Pascal theorem.

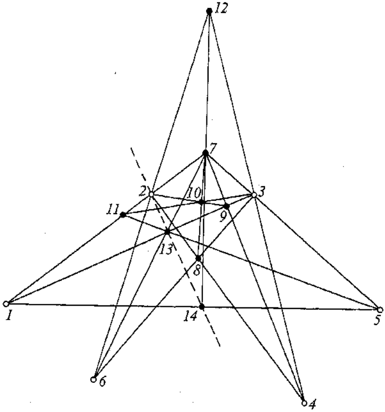

Fig. 51

Bibliography

1. Hilbert, D., and Cohn-Vassen, S. *Geometry and the Imagination.* Translated by P. Nemenyi. New York: Chelsea Publishing Company, 1952.
2. Courant, R., and Robbins, H. *What Is Mathematics?* London: Oxford University Press, 1941.
3. Pickert, G. *Projective Ebenen.* Berlin: Springer, 1955.
4. Coxeter, H. S. M. *The Real Projective Plane,* 2d ed. Cambridge: Cambridge University Press, 1955.

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TOPICS IN MATHEMATICS is a series of booklets translated and adapted from the Russian series *Popular Lectures in Mathematics*. These American editions have been prepared by the Survey of Recent East European Mathematical Literature at the University of Chicago under a grant from the National Science Foundation. These booklets provide students of mathematics at various levels, as well as other interested readers, with valuable supplementary material to further their mathematical knowledge and development.

This booklet, CONFIGURATION THEOREMS, presents several interesting theorems on collinear points and concurrent lines, with their applications to a number of practical geometric problems. In doing this, the booklet introduces the reader to certain fundamental concepts of projective geometry. Only the most elementary background in plane and solid geometry is required.

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